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Self-similar solution of a tensile crack problem in a coupled formulation☆

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ABSTRACT

Using a self-similar variables, an asymptotic investigation is carried out into the stress fields and the rates of creep deformations and degree of damage close to the tip of a tensile crack under creep conditions in a coupled (creep - damage) plane formulation of the problem. It is shown that a domain of completely damaged material (DCDM) exists close to the crack tip. The geometry of this domain is determined for different values of the material parameters appearing in the constitutive relations of the Norton power law in the theory of steady-state creep and a kinetic equation which postulates a power law for the damage accumulation. It is shown that, if the boundary condition at the point at infinity is formulated as the condition of asymptotic approximation to the Hutchinson–Rice-Rosengren solution [Hutchinson JW. Singular behaviour at the end of a tensile crack in a hardening material. *J Mech Phys Solids* 1968;**16**(1):13–31; Rice JR, Rosengren GF. Plane strain deformation near a crack tip in a power-law hardening material. *J Mech Phys Solids*. 1968;**16**(1):1–12], then the boundaries of the DCDM, which are defined by means of binomial and trinomial expansions of the continuity parameter, are substantially different with respect to their dimension and shape. A new asymptotic of the for stress field, which determines the geometry of the DCDM and leads to close configurations of the DCDM constructed using binomial and trinomial asymptotic expansions of the continuity parameter, are established by an asymptotic analysis and a numerical solution of the non-linear eigenvalue problem obtained.

An approach, similar to that used below, was implemented earlier to investigate a problem of an antiplane shear crack under creep conditions in a medium with damage.[4] It was shown that a domain of completely damaged material exists in the neighbourhood of the tip of a steady crack (this can be interpreted as a domain, adjacent to the crack surfaces, occupied by microcracks which are orthogonally orientated with respect to the main (macro) crack). In this domain, all the components of the stress tensor are zero and the continuity parameter reaches a critical value (in the given case, it is assumed that this value is equal to zero).

The study of the stress-strain state and the continuity distribution in the neighbourhood of the tip of a tensile crack under conditions of a plane strain state and a plane stress state is a natural continuation of investigations in this area.

1. Formulation of the problem

A neighbourhood of a semi-infinite tensile crack in an unbounded body is considered. The constitutive relations of the material of the body are constructed using Norton's power law of the theory of steady creep

$$\dot{\varepsilon}_{ij} = \frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{s_{ij}}{\psi} \tag{1.1}$$

when

$$s_{rr} = -s_{\theta\theta} = (\sigma_{rr}-\sigma_{\theta\theta})/2, \quad \sigma_e^2 = 3(\sigma_{rr}-\sigma_{\theta\theta})^2/4 + 3\sigma_{r\theta}^2$$

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in the case of a plane strain state and

$$s_{rr} = (2\sigma_{rr} - \sigma_{\theta\theta})/3, \quad s_{\theta\theta} = (2\sigma_{\theta\theta} - \sigma_{rr})/3, \quad \sigma_e^2 = \sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2$$

in the case of a plane stress state. Here, $\dot{\varepsilon}_{ij}$ are the components of the rate of creep deformation tensor, B and n are constants of the material, σ_e is the stress intensity, σ_{ij} are the components of the stress tensor, ψ is the Kachanov continuity parameter[5] ($1-\psi$ is the Rabotnov damage parameter[6]) and $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ are the components of the stress deviator.

In a polar system of coordinates with the pole at the crack tip, we have the equilibrium equations

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} = 0 \tag{1.2}$$

the condition for the compatibility of the deformations, formulated for the rates of creep deformation

$$2\frac{\partial}{\partial r}\left(r\frac{\partial\dot{\varepsilon}_{r\theta}}{\partial\theta}\right) = \frac{\partial^2\dot{\varepsilon}_{rr}}{\partial\theta^2} - r\frac{\partial\dot{\varepsilon}_{rr}}{\partial r} + r\frac{\partial^2(r\dot{\varepsilon}_{\theta\theta})}{\partial r^2} \tag{1.3}$$

and the kinetic equation, which postulates a power law for the accumulation of damage

$$\frac{d\psi}{dt} = -A\left(\frac{\sigma_{\text{eqv}}}{\psi}\right)^m; \quad \sigma_{\text{eqv}} = \alpha\sigma_e + \beta\sigma_1 + (1-\alpha-\beta)\sigma_{kk} \tag{1.4}$$

where A and m are material constants, t is the time, σ_{eqv} is the equivalent stress, σ_1 is the maximum principal stress, σ_{kk} is the hydrostatic stress, and the constants α and β are found experimentally.

The constitutive relations (1.1) are represented in the form: in the case of a plane strain state

$$\dot{\varepsilon}_{rr} = -\dot{\varepsilon}_{\theta\theta} = \frac{3}{4}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{rr} - \sigma_{\theta\theta}}{\psi}, \quad \dot{\varepsilon}_{r\theta} = \frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi} \tag{1.5}$$

and in the case of a plane stress state

$$\dot{\varepsilon}_{rr} = \frac{1}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{rr} - \sigma_{\theta\theta}}{\psi}, \quad \dot{\varepsilon}_{\theta\theta} = \frac{1}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{2\sigma_{\theta\theta} - \sigma_{rr}}{\psi}$$

$$\dot{\varepsilon}_{r\theta} = \frac{3}{2}B\left(\frac{\sigma_e}{\psi}\right)^{n-1}\frac{\sigma_{r\theta}}{\psi} \tag{1.6}$$

The conditions that there are no surface stresses on the crack surfaces have the form

$$\sigma_{\theta\theta}(r, \theta = \pm\pi, t) = 0, \quad \sigma_{r\theta}(r, \theta = \pm\pi, t) = 0 \tag{1.7}$$

In the case of constitutive relations (1.1), the boundary condition at the point at infinity can be formulated as the requirement that there is an asymptotic approximation to the Hutchinson–Rice-Rosengren (HRR) solution[2,3]

$$\sigma_{ij}(r \to \infty, \theta, t) = (C^*/(BI_n r))^{1/(n+1)}\bar{\sigma}_{ij}(\theta, n) \tag{1.8}$$

where C^* is a fracture parameter, defined by a contour integral, which is independent of the integration path, and was introduced for the analysis of bodies with cracks under creep conditions, I_n is a function which depends on n and is defined as a dimensionless C^*-integral and $\bar{\sigma}_{ij}(\theta, n)$ are functions which are known from the HRR solution.

2. Self-similar variable in the problem of a crack in a medium with damage

In the case of a stationary or growing semi-infinite crack in an infinite body in a material with the constitutive relations of the coupled problem in creep and damage mechanics theory, constructed using a power relation between the rates of creep deformation and the stresses (1.1), the initial conditions have the form

$$\sigma_{ij}(r, \theta, t = 0) = (C^*/(BI_n r))^{1/(n+1)}\bar{\sigma}_{ij}(\theta, n) \tag{2.1}$$

The asymptotic condition when $r \to \infty$ is determined by the solution of the analogous problem without taking account of the damage accumulation ($\psi \equiv 1$). It should be noted that the initial conditions when $t = 0$ (2.1) and the boundary condition at the point at infinity (1.8) are identical, since they are specified by the solution of the problem for $\psi \equiv 1$. It should be emphasized that the asymptotic condition (1.8) is a hypothesis, according to which the stress fields remote from the crack tip and close to the tip of a stationary crack in a material with power-law constitutive relations are identical. It has been established[1] that a self-similar variable

$$R = (Br/C^*)(At)^{-(n+1)/m} \tag{2.2}$$

exists for the constitutive relations (1.1) with initial and boundary conditions (2.1) and (1.8).

Expression (2.2) and the existence of the self-similar variable R are easily proved using dimensional analysis. Introduction of the dimensionless quantities

$$\hat{r} = r/L, \quad \hat{t} = t/T, \quad \hat{\sigma}_{ij} = \sigma_{ij}(C^*/(BL))^{-1/(n+1)}$$

where L is a certain characteristic length, T is a characteristic time, and the characteristic length and time can be related by an analysis of the kinetic damage accumulation equation (1.4) and $T=A^{-1}C^*/(BL)^{-m/(n+1)}$ enables one to represent the dimensionless stresses $\hat{\sigma}_{ij}$ as functions of the dimensionless variables in the form

$$\hat{\sigma}_{ij}(\hat{r}, \theta, \hat{t}) = (C^*/(BL))^{-1/(n+1)}\sigma_{ij}(r/L, \theta, tA(C^*/(BL))^{m/(n+1)}) \tag{2.3}$$

Since there is no characteristic linear dimension L in the problem being considered, it has to be eliminated from the arguments of the function $\hat{\sigma}_{ij}$, which is achieved by introducing the self-similar variable

$$R = \frac{r/L}{[tA(C^*/(BL))^{m/(n+1)}]^{(n+1)/m}} \tag{2.4}$$

As a result, the self-similar variable (2.2) can be obtained. In this case, the stresses and the continuity parameter are represented in the form

$$\sigma_{ij}(r, \theta, t) = (At)^{-1/m}\hat{\sigma}_{ij}(R, \theta), \quad \psi(r, \theta, t) = \hat{\psi}(R, \theta) \tag{2.5}$$

where $\hat{\sigma}_{ij}(R, \theta)$ and $\hat{\psi}(R, \theta)$ are dimensionless functions of the dimensionless variables R and θ and are to be determined when solving the boundary-value problems.

It should be noted that the boundary condition at the point at infinity can be formulated in a more general form compared with (1.8), namely

$$\sigma_{ij}(r \to \infty, \theta, t) \to \tilde{C}r^s\bar{\sigma}_{ij}(\theta, n) \tag{2.6}$$

The exponent s is to be determined when solving the problem, and $\tilde{C}$ is the amplitude of the stress field at infinity, which is determined by the geometry of the real sample and the system of applied loads. The self-similar variable for the boundary condition at the point at infinity (2.6) is introduced by a method, similar to that described earlier, and it is based on an analysis of the dimensions of the quantities appearing in the problem. Actually, the self-similar variable

$$R = r[tA\tilde{C}^m]^{1/(sm)} \tag{2.7}$$

exists in the case of the power constitutive relations (1.1) with the boundary conditions at the point at infinity (2.6).

The introduction of self-similar variable (2.7) does not change the form of equilibrium Eq. (1.2) and compatibility conditions (1.3), in which the coordinate r has to be replaced by the self-similar variable R. The kinetic equation and the boundary condition at the point at infinity undergo changes. The symbol over letters will henceforth omitted.

It is assumed from the results of investigations carried out earlier[7–10] that a domain of completely damaged material (DCDM) exists close to the crack tip, in which all the components of the stress tensor and the continuity parameter vanish. It is therefore necessary to find a solution of the system of equations consisting of equilibrium Eq. (1.2), compatibility conditions (1.3) and the kinetic equation

$$R\frac{\partial\psi}{\partial R} = -sm\left(\frac{\sigma_{eqv}}{\psi}\right)^m \tag{2.8}$$

over the whole of the domain $-\pi \le \theta \le \pi$ with the exception of the DCDM. The solution of this equation must satisfy the following boundary conditions:

the conditions that there are no surface stresses on the upper surface of the crack

$$\sigma_{\theta\theta}(R, \theta = \pi) = 0, \quad \sigma_{R\theta}(R, \theta = \pi) = 0 \tag{2.9}$$

the symmetry conditions in its continuation

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta}(R, \theta = 0) = 0, \quad \sigma_{R\theta}(R, \theta = 0) = 0 \tag{2.10}$$

and the asymptotic condition at the point at infinity

$$\sigma_{ij}(R \to \infty, \theta) \sim R^s\tilde{\sigma}_{ij}(\theta, n) \;\; s < 0 \tag{2.11}$$

3. Asymptotic solution

3.1. Plane strain state

The power form of the constitutive relations of the problem and the kinetic damage accumulation equation enables us to find a binomial asymptotic expansion of the Airy stress function and a trinomial expansion of the continuity parameter for large distances from the crack tip in the form

$$F(R,\theta) = R^{\lambda}f_0(\theta) + R^{\lambda_1}f_1(\theta) + o(R^{\lambda_1})$$
$$\psi(R,\theta) = 1 - R^{\gamma}g_0(\theta) - R^{\gamma_1}g_1(\theta) + o(R^{\gamma_1}) \quad (3.1)$$

where λ, λ_1, λ, λ_1 and $f_0(\theta), f_1(\theta), g_0(\theta), g_1(\theta)$ are unknown eigenvalues and eigenfunctions, to be determined. The binomial asymptotic expansions of the components of the stress tensor and the stress intensity when $R \to \infty$ are then defined by the relations

$$\sigma_{RR}(R,\theta) = R^s\sigma_{RR}^{(0)} + R^{s_1}\sigma_{RR}^{(1)} + \ldots = R^s(\lambda f_0 + f_0'') + R^{s_1}(\lambda_1 f_1 + f_1'') + \ldots$$
$$\sigma_{\theta\theta}(R,\theta) = R^s\sigma_{\theta\theta}^{(0)} + R^{s_1}\sigma_{\theta\theta}^{(1)} + \ldots = R^s\lambda(\lambda-1)f_0 + R^{s_1}\lambda_1(\lambda_1-1)f_1 + \ldots$$
$$\sigma_{R\theta}(R,\theta) = R^s\sigma_{R\theta}^{(0)} + R^{s_1}\sigma_{R\theta}^{(1)} + \ldots = -R^s(\lambda-1)f_0' - R^{s_1}(\lambda-1_1)f_1' + \ldots$$
$$\sigma_e(R,\theta) = R^s\sigma_e^{(0)}(\theta) + R^{s_1}\sigma_e^{(1)}(\theta)/\sigma_e^{(0)}(\theta) + \ldots; \quad s = \lambda - 2, \quad s_1 = \lambda_1 - 2 \quad (3.2)$$

The coefficients $\sigma_e^{(0)}(\theta)$ and $\sigma_e^{(1)}(\theta)$ are calculated from the formulae

$$\sigma_e^{(0)}(\theta) = \frac{\sqrt{3}}{2}\sqrt{(\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)})^2 + 4(\sigma_{R\theta}^{(0)})^2}$$

$$\sigma_e^{(1)}(\theta) = (\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)})(\sigma_{RR}^{(1)} - \sigma_{\theta\theta}^{(1)}) + 4\sigma_{R\theta}^{(0)}\sigma_{R\theta}^{(1)}$$

An asymptotic analysis of the kinetic equation enables us to conclude that $\gamma = sm$, $\gamma_1 = sm + s_1 - s$ (the hypothesis that the orders of smallness of the principal terms of the asymptotic expansions of the terms on the right and left-hand sides of the kinetic equation are the same is adopted here). The constitutive relations of the problem lead to the following asymptotic expansions of the rate of creep deformation at a considerable distance from the crack tip. For example:

$$\dot{\varepsilon}_{RR} = R^{sn}(\sigma_e^{(0)})^{n-1}\{1 + R^{sm}[(n-1)\sigma_e^{(1)}(\sigma_e^{(0)})^{-2} + ng_0]\} \times$$
$$\times [(\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)}) + R^{s_1 - s}(\sigma_{RR}^{(1)} - \sigma_{\theta\theta}^{(1)})] \quad (3.3)$$

It follows from equality (3.3) that the principal term of the asymptotic expansion is of the order of R^{sn} and the terms following it are of the order of R^{sn+sm} and R^{sn+s_1-s}. It should be noted that, by considering the coefficients of the principal terms of the asymptotic expansions in the compatibility condition and the damage accumulation law, it is possible to arrive at an ordinary differential equation (ODE) for determining the function $f_0(\theta)$ and, then, an algebraic equation which enables one to find the function $g_0(\theta)$. In investigating the terms of higher orders of smallness, it is necessary to obtain the ODE in order to find the function $f_i(\theta)$, assuming that the functions $f_0(\theta), f_1(\theta), \ldots, f_{i-1}(\theta)$ and $g_0(\theta), g_1(\theta), \ldots, g_{i-1}(\theta)$ have been previously determined. The construction of the asymptotic expansions and the availability of the chain of ODEs for finding the coefficients of these expansions lead to the condition $s_1 = s + sm$, which enables all the terms in the asymptotic expansion of the rate of creep deformation (3.3) to be taken into account. Consequently, the binomial asymptotic expansions of the rates of creep deformation take the form

$$\dot{\varepsilon}_{RR} = -\dot{\varepsilon}_{\theta\theta} = R^{sn}\varepsilon_{RR}^{(0)} + R^{s(n+m)}\varepsilon_{RR}^{(1)}, \quad \dot{\varepsilon}_{R\theta} = R^{sn}\varepsilon_{R\theta}^{(0)} + R^{s(n+m)}\varepsilon_{R\theta}^{(1)} \quad (3.4)$$

where

$$\varepsilon_{RR}^{(0)} = (\sigma_e^{(0)})^{n-1}(\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)}), \quad \varepsilon_{R\theta}^{(0)} = 2(\sigma_e^{(0)})^{n-1}\sigma_{R\theta}^{(0)}$$

$$\varepsilon_{RR}^{(1)} = (\sigma_e^{(0)})^{n-1}[\sigma_{RR}^{(1)} - \sigma_{\theta\theta}^{(1)} + \omega^{(1)}(\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)})]$$

$$\varepsilon_{R\theta}^{(1)} = 2(\sigma_e^{(0)})^{n-1}[\sigma_{R\theta}^{(1)} + \omega^{(1)}\sigma_{R\theta}^{(0)}]$$
$$\omega^{(1)} = (n-1)\sigma_e^{(1)}(\sigma_e^{(0)})^{-2} + ng_0$$

Substitution of the binomial asymptotic expansions of the components of the rate of creep deformation tensor (3.4) into the compatibility condition leads to two ODEs in functions of the angular coordinate

$$2(sn+1)\frac{d\varepsilon_{R\theta}^{(0)}}{d\theta} = \frac{d^2\varepsilon_{RR}^{(0)}}{d\theta^2} - sn(sn+2)\varepsilon_{RR}^{(0)} \tag{3.5}$$

$$2[s(n+m)+1]\frac{d\varepsilon_{R\theta}^{(1)}}{d\theta} = \frac{d^2\varepsilon_{RR}^{(1)}}{d\theta^2} - s(n+m)[s(n+m)+2]\varepsilon_{RR}^{(1)} \tag{3.6}$$

Eq. (3.5) leads to a fourth order non-linear ODE in the function $f_0(\theta)$

$$\begin{aligned}[n\Phi^2 + 4(\lambda-1)^2 f_0'^2]f_0^{IV} &= \kappa[(n-1)K(\theta)f_0' + (\sigma_e^{(0)})^2 f_0''] - \\ &- (n-1)(n-3)[K(\theta)/\sigma_e^{(0)}]^2\Phi - (n-1)[M(\theta)\Phi + 2K(\theta)\Phi'] - \\ &- \lambda(2-\lambda)(\sigma_e^{(0)})^2 f_0'' + sn(sn+2)(\sigma_e^{(0)})^2\Phi\end{aligned} \tag{3.7}$$

where

$$\Phi = \lambda(2-\lambda)f_0 + f_0'', \quad \lambda = -4(sn+1)(\lambda-1), \quad K(\theta) = \Phi\Phi' + 4(\lambda-1)^2 f_0' f_0''$$

$$M(\theta) = \lambda(2-\lambda)f_0''\Phi + \Phi'^2 + 4(\lambda-1)^2(f_0''^2 + f_0' f_0''')$$

The solution of Eq. (3.7) must satisfy the boundary conditions following from the requirement that there are no surface stresses on the upper surface of the crack and the symmetry conditions for its continuation:

$$f_0(\theta=\pi) = 0, \quad f_0'(\theta=\pi) = 0, \quad f_0'(\theta=0) = 0, \quad f_0'''(\theta=0) = 0 \tag{3.8}$$

A binomial boundary value problem is therefore formulated for determining the function $f_0(\theta)$, the solution of which was found numerically using the Runge–Kutta-Fehlberg (RKF) and shooting methods. Eq. (3.7) is homogeneous, and the normalization condition for the solution $f_0(\theta=0)=1$ is therefore adopted. When using the RKF method, it is necessary to select a constant $f_0''(\theta=0)=A_1$ and an eigenvalue λ such that the boundary conditions on the upper surface of the crack (3.8) are satisfied.

After determining the principal term of the asymptotic expansion of the Airy stress function, the kinetic equation enables us to find the binomial asymptotic expansion of the continuity parameter. The kinetic Eq. (2.8), and substitution of the asymptotic expansions (3.1) into it, lead to the asymptotic equality

$$\gamma R^{\gamma} g_0(\theta) + \gamma_1 R^{\gamma_1} g_1(\theta) = smR^{sm}(\sigma_{\text{eqv}}^{(0)})^m[1 + mR^{sm}(\sigma_{\text{eqv}}^{(1)}/\sigma_{\text{eqv}}^{(0)} + g_0)]$$

where

$$\sigma_{\text{eqv}}^{(0)} = (3-3\alpha-2\beta)(\sigma_{RR}^{(0)} + \sigma_{\theta\theta}^{(0)})/2 + (\alpha + \beta/\sqrt{3})\sigma_e^{(0)}$$

$$\sigma_{\text{eqv}}^{(1)} = (3-3\alpha-2\beta)(\sigma_{RR}^{(1)} + \sigma_{\theta\theta}^{(1)})/2 + (\alpha + \beta/\sqrt{3})\sigma_e^{(1)}/\sigma_e^{(0)}$$

from which the coefficients of the trinomial asymptotic expansion of the continuity parameter

$$g_0(\theta) = (\sigma_{\text{eqv}}^{(0)})^m; \quad g_1(\theta) = m(\sigma_{\text{eqv}}^{(0)})^m(\sigma_{\text{eqv}}^{(1)}/\sigma_{\text{eqv}}^{(0)} + g_0)/2 \tag{3.9}$$

can be found by equating the coefficients of like powers of R.

The binomial asymptotic expansion of the continuity parameter and the compatibility condition (Eq. (3.6)) lead to an inhomogeneous linear ODE in the function $f_1(\theta)$, which is investigated numerically taking account of the boundary conditions on the upper surface of the crack and the symmetry conditions for its continuation, which are analogous to conditions (3.8) for the function $f_0(\theta)$.

To use the RKF method, it is necessary to formulate the following initial conditions when $\theta=0$

$$f_1(\theta=0) = A_2, \quad f_1'(\theta=0) = 0, \quad f_1''(\theta=0) = A_3, \quad f_1'''(\theta=0) = 0$$

The constants A_2 and A_3 are found when solving the boundary value problem for the function $f_1(\theta)$. After determining the function $f_1(\theta)$ and, consequently, also $\sigma_{\text{eq}\nu}^{(1)}$, it is possible to find the function $g_1(\theta)$ using the second equality of (3.9). The configuration of the DCDM is therefore determined by the relations

$$\psi = 1 - R^{sm}g_0(\theta) = 0 \quad \text{or} \quad R(\theta) = [g_0(\theta)]^{-1/(sm)} \tag{3.10}$$

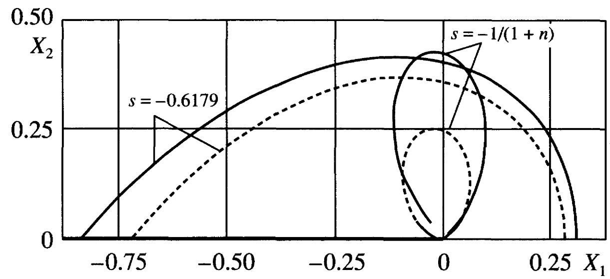

Fig. 1.

$$\psi = 1 - R^{sm}g_0(\theta) - R^{2sm}g_1(\theta) = 0 \quad \text{or}$$
$$R(\theta) = \left\{g_0(\theta)/2 + \sqrt{[g_0(\theta)/2]^2 + g_1(\theta)}\right\}^{-1/(sm)} \tag{3.11}$$

and formulae (3.10) (formulae (3.11) enable us to find the boundary of the DCDM, which is determined by the binomial (trinomial) asymptotic expansion of the continuity parameter.

The boundaries of the DCDM, found using the binomial expansion (the dashed curve) and the trinomial expansion (the solid curve) of the continuity parameter for $s = -1/(n+1)$ and $n = 5$, which corresponds to the asymptotic condition of convergence to the HRR solution and the more general boundary condition (2.11), are shown in Fig. 1.

In all the figures $m = 0.7n$ and, by virtue of the symmetry, only the domain $X_2 \geq 0$ is shown. The crack surfaces are represented by a bold line.

It can be seen that, if $s = -1/(n+1)$ is taken, then the dimensions of the DCDM, which are determined by the binomial and trinomial asymptotic expansions of the continuity parameter, are considerably different since the characteristic linear dimension of the DCDM (the length of the domain along the ordinate axis) increases significantly when account is taken of the third term in (3.1) which no longer serves as a small correction to the principal term of the asymptotic representation (3.1). A similar pattern also occurs for other values of n.

The boundary condition at the point at infinity therefore cannot be formulated as a condition for the asymptotic convergence with the HRR solution, and, in subsequent calculations, the boundary condition at infinity was therefore adopted in the more general form of (2.11) and the eigenvalue s was determined numerically. The values of s found, that lead to DCDMs, which are determined by means of the binomial and trinomial asymptotic expansions of the continuity parameter, and are close in their shape and characteristic dimension, are shown below

n	3	4	5	6	7	8	9
s	−0.77169	−0.66848	−0.61790	−0.59012	−0.57324	−0.56213	−0.55436

The values of s obtained, which appear in boundary conditions (2.11), lead to a new asymptotic of the far stress field in the coupled (creep - damage) problem. The values of the constants $f_0''(0)$, $f_1(0)$ and $f_1''(0)$, which are determined during the course of the analysis, are shown below

n	3	4	5	6	7	8	9
$-f_0''(0)\cdot 10^4$	4372.4	4092.0	3985.5	3950.2	3943.7	3948.6	3958.1
$f_1(0)\cdot 10^4$	1036.91	363.97	122.58	42.069	14.862	5.396	2.005
$f_1''(0)\cdot 10^4$	−471.07	−109.09	19.2	36.92	27.41	16.44	0.89

The effect of the critical value of the continuity parameter ψ_{cr} is represented in Fig. 2 for $n = 8$. The boundaries of the domain of damaged material are shown for different critical values of the continuity parameter. A change in the critical value of the continuity parameter does not lead to a substantial change in the DCDM.

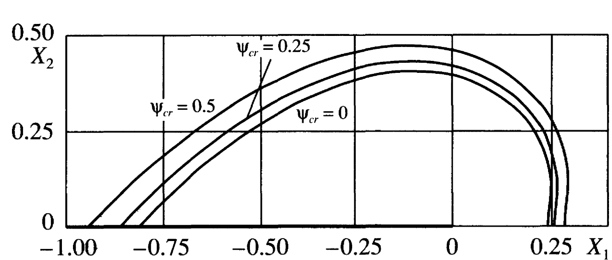

Fig. 2.

3.2. Plane stress state

In the case of a plane stress state, the binomial asymptotic expansions of the components of the stress tensor are also determined by equalities (3.2) but the coefficients $\sigma_e^{(0)}(\theta)$ and $\sigma_e^{(1)}(\theta)$ are calculated using the formulae

$$\sigma_e^{(0)}(\theta) = \sqrt{(\sigma_{RR}^{(0)})^2 + (\sigma_{\theta\theta}^{(0)})^2 - \sigma_{RR}^{(0)}\sigma_{\theta\theta}^{(0)} + 3(\sigma_{R\theta}^{(0)})^2}$$

$$\sigma_e^{(1)}(\theta) = \sigma_{RR}^{(0)}\sigma_{RR}^{(1)} + \sigma_{\theta\theta}^{(0)}\sigma_{\theta\theta}^{(1)} - \sigma_{RR}^{(0)}\sigma_{\theta\theta}^{(1)} - \sigma_{RR}^{(1)}\sigma_{\theta\theta}^{(0)} + 3\sigma_{R\theta}^{(0)}\sigma_{R\theta}^{(1)}$$

The asymptotic representation of the components of the creep rate tensor for large R has the form

$$\dot\varepsilon_{RR} = R^{sn}\varepsilon_{RR}^{(0)} + R^{s(n+m)}\varepsilon_{RR}^{(1)}, \quad \dot\varepsilon_{\theta\theta} = R^{sn}\varepsilon_{\theta\theta}^{(0)} + R^{s(n+m)}\varepsilon_{\theta\theta}^{(1)}$$

$$\dot\varepsilon_{R\theta} = R^{sn}\varepsilon_{R\theta}^{(0)} + R^{s(n+m)}\varepsilon_{R\theta}^{(1)} \tag{3.12}$$

where

$$\varepsilon_{RR}^{(0)} = (\sigma_e^{(0)})^{n-1}(2\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)}), \quad \varepsilon_{\theta\theta}^{(0)} = (\sigma_e^{(0)})^{n-1}(2\sigma_{\theta\theta}^{(0)} - \sigma_{RR}^{(0)}), \quad \varepsilon_{R\theta}^{(0)} = 3(\sigma_e^{(0)})^{n-1}\sigma_{R\theta}^{(0)}$$

$$\varepsilon_{RR}^{(1)} = (\sigma_e^{(0)})^{n-1}[2\sigma_{RR}^{(1)} - \sigma_{\theta\theta}^{(1)} + \omega^{(1)}(2\sigma_{RR}^{(0)} - \sigma_{\theta\theta}^{(0)})]$$

$$\varepsilon_{\theta\theta}^{(1)} = (\sigma_e^{(0)})^{n-1}[2\sigma_{\theta\theta}^{(1)} - \sigma_{RR}^{(1)} + \omega^{(1)}(2\sigma_{\theta\theta}^{(0)} - \sigma_{RR}^{(0)})]$$

$$\varepsilon_{R\theta}^{(1)} = 3(\sigma_e^{(0)})^{n-1}[\sigma_{R\theta}^{(1)} + \omega^{(1)}\sigma_{R\theta}^{(0)}]$$

By substituting the binomial asymptotic expansions of the rate of deformation tensor (3.12) into the condition for the compatibility of the rates of deformation (1.3) and equating the coefficients of like powers of R, it is possible to find two ODEs for determining the angular distributions of the stress tensor components and the continuity parameter

$$2(sn+1)\frac{d\varepsilon_{R\theta}^{(0)}}{d\theta} = \frac{d^2\varepsilon_{RR}^{(0)}}{d\theta^2} - sn\varepsilon_{RR}^{(0)} + sn(sn+1)\varepsilon_{\theta\theta}^{(0)}$$

$$2[s(n+m)+1]\frac{d\varepsilon_{R\theta}^{(1)}}{d\theta} = \frac{d^2\varepsilon_{RR}^{(1)}}{d\theta^2} - s(n+m)\varepsilon_{RR}^{(1)} + sn[s(n+m)+1]\varepsilon_{\theta\theta}^{(1)}$$

and, consequently, the function $f_0(\theta)$ satisfies the ODE

$$[(n-1)\Psi^2/2 + 2(\sigma_e^{(0)})^2]f_0^{\mathrm{VI}} = \kappa[(n-1)K(\theta)f_0' + (\sigma_e^{(0)})^2 f_0''] -$$

$$-(n-1)(n-3)[K(\theta)/\sigma_e^{(0)}]^2\Psi - (n-1)[M(\theta)\Psi + 2K(\theta)\Psi'] -$$

$$-\lambda(3-\lambda)(\sigma_e^{(0)})^2 f_0'' + sn(\sigma_e^{(0)})^2\Psi - sn(sn+1)(\sigma_e^{(0)})^2\chi \tag{3.13}$$

where

$$\Psi = \lambda(3-\lambda)f_0 + 2f_0'', \quad \kappa = -6(sn+1)(\lambda-1), \quad \chi = \lambda(2\lambda-3)f_0 - f_0''$$

$$K(\theta) = [(\lambda f_0' + f_0''')\Psi + \lambda(\lambda-1)f_0'\chi + 6(\lambda-1)^2 f_0' f_0'']/2$$

$$M(\theta) = [(\lambda f_0' + f_0''')\Psi' + \lambda(\lambda-1)f_0'\chi' + \lambda f_0''\Psi +$$

$$+\lambda(\lambda-1)f_0''\chi + 6(\lambda-1)^2(f_0''^2 + f_0' f_0''')]/2$$

The solution of the non-linear ODE (3.13), which satisfies the conditions

$$f_0(\theta=0) = 1, \quad f_0'(\theta=0) = 0, \quad f_0''(\theta=0) = A_1, \quad f_0'''(\theta=0) = 0$$

was sought numerically in a similar way to the case of the plane strain state described above. A boundary-value problem in the function $f_1(\theta)$ is then formulated, which enables a trinomial asymptotic expansion of the continuity parameter to be constructed and enables the configuration of the DCDM to be constructed in the case of a plane stress state.

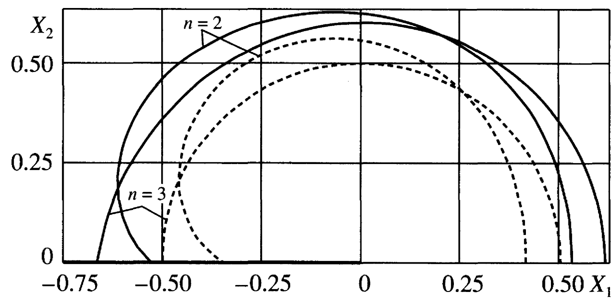

Fig. 3.

Proceeding as in the case of a plane strain state, it can be concluded that the coefficients of the trinomial asymptotic expansion of the continuity parameter are determined by expressions (3.9), but the functions appearing in them are found using the formulae

$$\sigma_{\text{eqv}}^{(0)}(\theta) = (1-\alpha-\beta/2)(\sigma_{RR}^{(0)}+\sigma_{\theta\theta}^{(0)})+\alpha\sigma_e^{(0)}+\beta p/2$$

$$\sigma_{\text{eqv}}^{(1)}(\theta) = (1-\alpha-\beta/2)(\sigma_{RR}^{(1)}+\sigma_{\theta\theta}^{(1)})+\alpha\sigma_e^{(1)}/\sigma_e^{(0)}+\beta p_1/(2p)$$

where

$$p^2 = \Phi^2+4(\lambda-1)^2 f_0'^2, \quad p_1 = [\lambda_1(2-\lambda_1)f_1+f_1'']\Phi+4(\lambda-1)(\lambda_1-1)f_0'f_1'$$

As in the case of a plane strain state, a comparison of the boundaries of the DCDM obtained using the binomial and trinomial expansions of the continuity parameter showed that the boundary condition at a point at infinity cannot be formulated as a condition of the convergence to the HRR solution and the boundary condition at an infinitely distant point is therefore taken in the more general form (2.6). The exponent s is to be determined when solving the eigenvalue problem for the non-linear ODE (3.13) with the boundary conditions

$$f_0(\theta=0) = 1, \quad f_0''(\theta=0) = A_1$$

and the conditions on the upper surface of the crack. It should be noted that the investigation of the eigenvalue problem obtained is of interest in its own right and is the subject of separate investigations (for example, an algorithm for the numerical solution of the problem has been described and the results of calculations for $n=3$ and $n=5$ have been presented[11]).

Within the framework of the present investigation, the search for the eigenvalues was performed numerically using the RKF method and the shooting method. The two parameters, the eigenvalue s and the constant A_1, were selected in order to satisfy the two boundary conditions on the upper surface of the crack. The values of s and A_1 were assumed to have been selected if the condition

$$\sqrt{[f_0(\pi)]^2+[f_0'(\pi)]^2} < \varepsilon$$

was satisfied, where ε is the specified accuracy of the calculations, which was taken as being equal to 10^{-5}. The eigenvalues s, found in the case of a plane stress state, are listed below

n	2	3	4	5	6	7	8	9
s	−1.15408	−1.0	−0.91338	−0.85801	−0.81979	−0.79195	−0.77084	−0.75432

The values of the constants $f_0''(0)$, $f_1(0)$ and $f_1''(0)$, obtained as the result of numerical calculation are given below

n	2	3	4	5	6	7	8	9
$-f_0''(0)\cdot 10^4$	5689.1	5000	4658.4	4428.6	4261.5	4134.6	4035.0	3955.4
$f_1(0)\cdot 10^4$	265.2	260.41	170.40	117.08	83.99	62.27	47.38	36.78
$f_1''(0)\cdot 10^4$	59.22	29.93	−73.86	−99.64	−102.38	−97.08	−88.99	−80.15

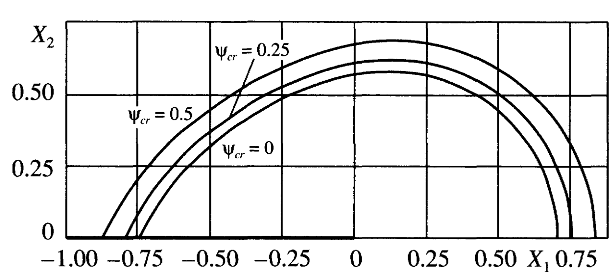

Fig. 4.

The results of the calculations show that the eigenvalues found for $n=3$ and $n=5$ are in close agreement with known results.[11] The configurations of the DCDM obtained using binomial and trinomial expansions of the continuity parameter are shown in Fig. 3 for $n=2$ and $n=3$. The boundaries of the DCDM, obtained using formulae (3.10) and (3.11), are shown by the dashed and solid curves. As in the case of a plane strain state, a change in the critical values of the continuity parameter does not lead to a substantial change in the DCDM (Fig. 4).

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References

1. Riedel H. *Fracture at High Temperature*. Berlin, etc.: Springer; 1987, 418 p.
2. Hutchinson JW. Singular behaviour at the end of a tensile crack in a hardening material. *J Mech Phys Solids* 1968;**16**(1):13–31.
3. Rice JR, Rosengren GF. Plane strain deformation near a crack tip in a power-law hardening material. *J Mech Phys Solids* 1968;**16**(1):1–12.
4. Stepanova LV, Fedina MYe. Self-similar solution of the problem of an antiplane shear crack in a coupled (creep - damage) formulation. *Zh Prikl Mekh Tekh Fiz* 2002;**43**(5):114–23.
5. Kachanov LM. Fracture time under creep conditions. *Izv Akad Nauk SSSR OTN* 1958:26–31.
6. Rabotnov YuN. The mechanism of long-term fracture. In: *Problems of the Strength of Materials and Structures*. Moscow: Izd. Akad. Nauk SSSR; 1959. p. 5–7.
7. Astafev VI, Grigorova TV, Pastukhov VA. Effect of material damage on the stress-strain state in the vicinity of the crack tip under creep. *Fiz-Khim Mekhanika Materialov* 1992;**28**(1):5–11.
8. Astafev VI, Grigorova TV. Stress and damage distribution at the tip of crack growing in a creep process. *Izv Ross Akad Nauk MTT* 1995;**3**:160–6.
9. Murakami S, Hirano T, Liu Y. Asymptotic fields of stress and damage of a mode I creep crack in steady-state growth. *Intern J Solids Structures* 2000;**37**(43):6203–20.
10. Zhao J, Zhang X. The asymptotic study of fatigue crack growth based on damage mechanics. *Engng Fract Mech* 1995;**50**(1):131–41.
11. Lu M, Lee SB. Eigenspectra and orders of singularity at a crack tip for a power-law creeping medium. *Intern J Fract* 1998;**92**(1):55–70.

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